International Review of Economics and Finance 24 (2012) 8–25

International Review of Economics and Finance

Implementing option pricing models when asset returns follow an autoregressive moving average process

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ARTICLE INFO

Article history:
Received 10 August 2010
Received in revised form 21 December 2011
Accepted 27 December 2011
Available online 4 January 2012

JEL classification:
G13
C52

Keywords:
ARMA process
Option pricing
Martingale

ABSTRACT

Motivated by the empirical findings that asset returns or volatilities are predictable, this paper studies the pricing of European options on stock or volatility, the instantaneous changes of which depend upon an autoregressive moving average (ARMA) process. The pricing formula of an ARMA-type option is similar to that of Black and Scholes, except that the total volatility input depends upon the AR and MA parameters. From numerical analyses, the option values are increasing functions of the levels of AR or MA parameters across all moneyness levels. Specifically, the AR effect is more significant than the MA effect. Finally, based on the daily closing prices of TAIEX options from 2004 to 2008, the ad hoc ARMA(1,1) model provides the best in-sample fit and the second best out-of-sample fit, whereas the variance gamma model provides the second best in-sample fit and the best out-of-sample fit. Therefore, both variance gamma model and ad hoc ARMA model are superior models for pricing TAIEX options.

1. Introduction

The pricing, hedging and risk management of derivatives is important because derivatives are now widely used to transfer risk in financial markets. At first, these options were priced and hedged using the classic Black–Scholes–Merton (BSM) assumptions. In particular, it was assumed that stock price returns follow a geometric Brownian motion; however, it has been well documented using empirical data that stock dynamics under the physical measure follow a more complicated process than the standard geometric Brownian motion. Hence, various extensions of the standard model have been proposed.

There is substantially empirical evidence for the predictability of financial asset returns. For example, Fama (1965) finds that the first-order autocorrelations of daily returns are positive for 23 of 30 Dow Jones Industrials. Fisher (1966) suggests that the autocorrelations of monthly returns on diversified portfolios are positive and larger than those for individual stocks. Gençay (1996) uses the daily Dow Jones Industrial Average Index from 1963 to 1988 to examine predictability of stock returns with buy–sell signals generated from the moving average rules. Conrad and Kaul (1988) also present positive autocorrelations of Wednesday-to-Wednesday returns for size-grouped portfolios of stocks. Chopra, Lakonishok, and Ritter (1992), De Bondt and Thaler (1985), French and Roll (1986), Jegadeesh (1990), and Lehmann (1990) all find negatively serial correlations in returns of individual stocks or various portfolios. Kittiakarasakun and Tse (2011) also obtain better results by evaluating VaR performance delivered by ARMA models using the index returns of several Asian stock markets. Consequently, the autoregressive moving average (ARMA) process is an alternative model to describe predictable asset returns.

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For modeling the predictable asset returns by distinguishing between the risk-neutral and physical distributions of an option's underlying asset return process, Grundy (1991) shows that, even though the underlying asset returns follow an Ornstein–Uhlenbeck (O–U) process, the BSM formula still holds. Lo and Wang (1995) construct an adjustment for predictability to the BSM formula and show that this adjustment can be important even for small levels of predictability. Liao and Chen (2006) derive the closed-form formula for a European option on an asset, the returns of which follow a continuous-time type of first-order moving average process. The pricing formula of these options is similar to that of BSM model, except for the volatility input. The first-order MA parameter is significant to option values even if the autocorrelation between asset returns is weak.

Considering the effect of infrequent trading, Jokivuolle (1998) assumes that the observed index, which is not the true index, follows a discrete-time ARIMA model and derives a pricing formula for European index options. Assuming that asset returns follow a discrete-time version of ARMA process, Huang, Wu, and Wang (2009a, 2009b) present a discrete-time ARMA model with least-square Monte Carlo (LSM) algorithm for valuing American options on stocks and then extend the discrete-time ARMA model for pricing TAIEX options. However, there are no studies theoretically deriving the pricing formula and then empirically testing it when asset returns follow a continuous-time type of generalized ARMA process. Thus, the main goal herein is to fill the gap.

In addition to financial asset returns, the future volatility of an asset's price can be forecasted using the ARMA model. Gallant, Hsu, and Tauchen (1999) show that the sum of two AR(1) processes is capable of capturing the persistent nature of asset price volatility. Alizadeh, Brandt, and Diebold (2002) show that the sum of two AR(1) processes describes FX volatility better than one AR(1) process. The sum of two AR(1) processes can always be represented by an ARMA(2,1) model (Granger & Newbold, 1976). Pong, Shackleton, Taylor, and Xu (2004) compare forecasts of the logarithms of realized volatility of the exchange rates. Their forecasts are obtained from an ARMA model, an ARFIMA model, a GARCH model and option implied volatilities. Using an MSE criterion, they find that the ARMA forecasts generally perform better than implied volatilities for short forecast horizons. The GARCH forecasts are the least accurate for most of the evaluations. Ferland and Lalancette (2006) evaluate volatility forecasts based on a multivariate loss function that matches as much as possible options on the Eurodollar trading on the CME. They considered four forecasting approaches: an ARMA, a feedforward neural network, a GARCH-diagonal-BEKK and a naive estimator based on previous-week realized estimates. They find that the ARMA-based and feedforward neural network based forecasts emerge as the best model-based forecasts. Motivated by the above empirical findings, the second goal of this study is to evaluate the European volatility option when the logarithms of volatility follow an ARMA process.

Based on the Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) option data from 2004 to 2008, a market ranked 3rd globally after KOSPI 200 options and Dow Jones Euro Stoxx 50 option contracts, the third goal of this study is to empirically test the continuous-time ARMA-type option pricing model. For model comparison, ad hoc BSM model of Dumas, Fleming, and Whaley (1998), variance gamma (VG) model of Madan, Carr, and Chang (1998) and GARCH(1,1) model of Heston and Nandi (2000) are employed as the competing models in the empirical analysis.

The remaining parts of this paper are organized as follows: Section 2 shows the setting of an ARMA process examined in this paper and provides the martingale property and the corresponding transformation in probability measure; Section 3 illustrates the pricing formula for the ARMA-type stock and volatility options; Sections 4 and 5 provide the results of numerical and empirical analyses, respectively. Section 6 is the conclusion.

2. Description of the model

In this section, an ARMA process, composed of the AR effect on the drift term and the MA effect on the diffusion term of the instantaneous stock return, is first described; consequently, an MA process used by Liao and Chen (2006) is the special case of the ARMA process with the AR order of zero and the MA order of one. The martingale measure that transforms the discounted stock price into a martingale is also defined.

2.1. An ARMA process of instantaneous asset returns

Without loss of generality, this paper denotes the underlying assets including dividends as S. The current time is t_0 and the expiration date of the options is T. Since the fact that stock returns follow an autoregressive moving average process and show significantly negative autocorrelations at very short lags in high-frequency data, this paper introduces an ARMA process and assumes the dynamics of the instantaneous asset return as follows:

$$d\ \ln S_t = \omega\, dt + \sum_{i=1}^{p} \alpha_i d\ \ln S_{t-ih} + \sum_{j=0}^{q} \sigma \beta_j dW^P_{t-jh}, \tag{1}$$

where p and q denote respectively the AR and MA orders, α_i and β_j are the AR and MA coefficients and $\beta_0 = 1$. ω is an arbitrary constant, $\sigma > 0$ is a constant volatility coefficient, $dt > 0$ is an infinitesimal time interval and $h > 0$ is a fixed but arbitrarily small constant. In addition, W^P_t is a one-dimension standard Brownian motion defined in a naturally filtered probability space $\left(\Omega, \mathfrak{I}, P, (\mathfrak{I}_t)_{t\in[0,T]}\right)$; dW^P_{t-ih}, $i = 1, \ldots, N$, are the instantaneous increments of the standard Brownian motion at time $t - ih$. For empirical work, h is restricted by the frequency of historical data. It is convenient to assume that $T \in [t_N, t_{N+1})$, where $t_n = t_0 + nh$, $n = 0, \ldots, N+1$. It is worth noting that Eq. (1) reduces to the continuous-time MA(1) process in Liao and Chen (2006) when the AR and MA coefficients are all zeros except for $\beta_0 = 1$ and $\beta_1 = \beta$.

Using the lag operators satisfying $L^k x_t = x_{t-kh}$, we have

$$\left(1-\alpha_1 L - \ldots - \alpha_p L^p\right) d\ln S_t = \omega dt + \sigma\left(1+\beta_1 L + \ldots + \beta_q L^q\right) dW_t^P, \tag{2}$$

Lemma 1. *When the roots of* $1-\alpha_1 z - \ldots - a_p z^P = 0$ *lie outside the unit circle, the stock price process S defined in Eq. (1) follows a stationary ARMA model, namely*

$$d\ln S_t = \mu dt + \sigma\psi(L) dW_t^P = \mu dt + \sigma \sum_{j=0}^{\infty} \psi j dW_{t-jh}^P, \tag{3}$$

where $\mu = \omega/(1-\alpha_1 - \ldots - \alpha_p)$. $\psi(L)$ *is given by*

$$\psi(L) = \sum_{j=0}^{\infty} \psi_j L^j = \frac{\left(1+\beta_1 L + \ldots + \beta_q L^q\right)}{\left(1-\alpha_1 L - \ldots - \alpha_p L^p\right)} \tag{4}$$

and satisfies $\sum_{j=0}^{\infty} \left|\psi_j\right| < \infty$. *In addition, the stock price process S which satisfies*

$$\ln S_t = \ln S_{t_0} + \mu(t-t_0) + \sigma \sum_{j=0}^{\infty} \int_{t_0-jh}^{t-jh} \psi j dW_u^P \tag{5}$$

is a unique strong solution of the stochastic differential equation (SDE) defined in Eq. (3) with initial condition S_{t_0}.

The proof of Lemma 1 is provided in Appendix A.

In view of Eq. (3), the variance of instantaneous stock returns at time t conditional on the time-t_0 information set is

$$Var_{t_0}(R_t) = \sigma^2 \left(\sum_{j=0}^{n} \left(\psi_j\right)^2\right) dt, \forall t \in [t_n, t_{n+1}) \tag{6}$$

and the conditional autocorrelation coefficient is

$$Corr_{t_0}\left(R_t, R_{t+gh}\right) = \frac{\sum_{j=o}^{n} \psi_j \psi_{j+g}}{\sqrt{\sum_{i=0}^{n} (\psi_i)^2} \sqrt{\sum_{i=0}^{n+g} (\psi_i)^2}}, \forall t \in [t_n, t_{n+1}) \tag{7}$$

where n is the integer part of $(t-t_0)/h$, $R_t \equiv d \ln S_t$ and g is positive integer. By virtue of the conditional variance and autocorrelation coefficient, the properties of stock returns specified in Eq. (1) can be observed. Apparently, the stock price process defined in Eq. (1), which reduces to a geometric Brownian motion when both AR and MA orders are zero, exhibits non-zero autocorrelation, the values of which depend upon the AR and MA coefficients.

Because the conditional variance and autocorrelation functions depend on the key parameters ψ_j, $j = 0, \ldots, \infty$, we provide the analytic formulas for ψ_j in Lemma 2.[1]

Lemma 2. *Assume that the stock price process S defined in Eq. (1) follows a stationary ARMA model. Let* $1-\alpha_1 L - \ldots \alpha_p L^p = (1-\lambda_1 L)(1-\lambda_2 L)\ldots(1-\lambda_p L)$, *where* $(\lambda_1, \lambda_2, \ldots, \lambda_p)$, *inside the unit circle, are the distinct eigenvalues of F which is defined as:*

$$F = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{p-1} & \alpha_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{8}$$

[1] Using MATLAB, we can directly obtain ψ_j by using the following codes: *LagOp*, *mrdivide* and *toCellArray*. For example, if A and B respectively denote $(1-0.5L)$ and $(1+0.15L+0.2L^2)$ for an ARMA(1,2) model with $\alpha_1 = 0.5$, $\beta_1 = 0.15$ and $\beta 2 = 0.2$, we can type A = *LagOp*({1 −0.5}), B = *LagOp*({10.15 0.2}) and ψ = *toCellArray*(*mrdivide*(B, A,'RelTol, 0,' Degree, 5)) in MATLAB command window. Then, ψ is equal to $[\psi_0, \ldots, \psi_5]$ = [1, 0.65, 0.525, 0.2625, 0.13125, 0.0656].

Applying the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_p)$, $j=0, \ldots, \infty$, *is given by*

$$\psi_j = \begin{cases} \sum_{i=1}^{p}\left(c_i\left(\sum_{k=0}^{j}\beta_{j-k}\lambda_i^k\right)\right), & \text{if } j<q \\ \sum_{i=1}^{p}\left(c_i\lambda_i^{j-q}\left(\sum_{k=0}^{q}\beta_{q-k}\lambda_i^k\right)\right), & \text{if } j\geq q, \end{cases} \tag{9}$$

where c_i, $i=1,\ldots,p$, *satisfying* $\sum_{i=1}^{p} c_i = 1$, *is of the form:*

$$c_i = \frac{\lambda_i^{p-1}}{\prod_{\substack{k=1,\ldots,p \\ k\neq i}}(\lambda_i-\lambda_k)}, \; i=1,\ldots,p. \tag{10}$$

The proof of Lemma 2 is provided in Appendix B. Note that, using Eq. (9), we can obtain $\psi_0 = \beta_0\sum_{i=1}^{p} c_i = 1$.

The underlying asset's log-price dynamics in the paper are similar to the discrete-time model used in Jokivuolle (1998) and Huang et al. (2009a,b) and the continuous-time one of Liao and Chen (2006). Specifically, Jokivuolle (1998) derives a pricing formula for European options on an observed stock index which is not the true index returns and follows an infinity-order moving average process. Huang et al. (2009a,b) present a discrete-time ARMA model with numerical method based LSM algorithm for valuing American options on stocks and then extend the model for pricing TAIEX options. However, unlike Jokivuolle (1998) and Huang et al. (2009a,b), this paper assumes that the process of asset returns is continuous-time type of ARMA process and the observed and true returns are identical. Furthermore, the MA(1) process used in Liao and Chen (2006) is a special case of the ARMA(p, q) process which is better to account for various types of empirical financial time-series data.

Two important modeling issues concerning Eq. (5) should be discussed. First, can the price process specified in Eq. (5) plausibly represent security price fluctuations? Second, does the price process specified in Eq. (5) rule out arbitrage opportunities? For the first issue, Harrison and Pliska (1981) demonstrate that, as long as the discounted price process is a martingale under risk neutral probability measure Q equivalent to P, the price process can be used to represent security price fluctuations. For the second issue, it is well known that there are no arbitrage opportunities if and only if risk neutral probability measure Q exists. For expository purposes, the preceding conditions will be checked in the next section.

2.2. Martingale property of an ARMA process

To price the financial derivatives written on a stock S, it is more convenient to have a risk-free security. Suppose the risk-free interest rate r is constant over the trading interval $[0,T]$ and the savings account, denoted by B, is assumed to be continuously compounded at rate r; that is,

$$dB_u = rB_u d\ u, \forall u\in[0,T]. \tag{11}$$

In other words, $B_t=e^{rt}$ with the usual convention that $B_0=1$. For $t\in[t_n, t_{n+1})$, the dynamics of the stock prices in Eq. (5) can be rewritten as follows:

$$\ln S_t = \ln S_{t_0} + r(t-t_0) - \frac{1}{2}V_n(t_0,t) + A_n(t_0,t) + Z_n^p(t_0,t), \forall t\in[t_n, t_{n+1}), \tag{12}$$

where

$$A_n(t_0,t) = (\mu-r)(t-t_0) + \frac{1}{2}V_n(t_0,t) + \sigma\sum_{j=n+1}^{\infty}\left(\psi_j\left(W_{t-jh}^P - W_{t_0-jh}^P\right)\right) + \sigma\sum_{i=0}^{n}\left(\psi_j\left(W_{t_0}^P - W_{t_0-ih}^P\right)\right), \quad \forall t\in[t_n, t_{n+1}), \tag{13}$$

$$Z_n^P(t_0,t) = \sigma\sum_{j=0}^{n}\left(\psi_j\left(W_{t-jh}^P - W_{t_0}^P\right)\right)\forall t\in[t_n, t_{n+1}), \tag{14}$$

$$V_n(t_0,t) \equiv Var_P\left(\ln\left(S_t/S_{t_0}\right)\middle|\Im_{t_0}\right)\forall t\in[t_n, t_{n+1}), \tag{15}$$

where n is the integer part of $(t-t_0)/h$. Rearranging $Z_n^P(t_0, t)$ as the sum of independent increments of Brownian motion, we obtain

$$\begin{aligned} Z_n^P(t_0,t) &= \sigma\left[\psi_0\int_{t_0}^{t} dW_u^P + \psi_1\int_{t_0}^{t-h} dW_u^P + \dots\psi_n\int_{t_0}^{t-nh} dW_u^P\right] \\ &= 1_{(n>0)}\left(\sigma\sum_{j=0}^{n-1}\left(\left(\sum_{i=0}^{j}\psi_i\right)\int_{t-(j+1)h}^{t-jh} dW_u^P\right)\right) + \sigma\left(\sum_{i=0}^{n}\psi_i\right)\int_{t_0}^{t-nh} dW_u^P. \end{aligned} \tag{14'}$$

Therefore, the conditional variance of log return obtained by using the independent increments property of Brownian motion is given by

$$V_n(t_0,t) \equiv Var_P\left(\ln\left(S_t/S_{t_0}\right)\middle|\Im_{t_0}\right) = \sigma^2\left(1_{(n>0)}\sum_{j=0}^{n-1}\left(\sum_{i=0}^{j}\psi_i\right)^2 h + \left(\sum_{i=0}^{n}\psi_i\right)^2 (t-t_n)\right). \tag{15'}$$

Conditioning on $\Im_{t_0}$, the paths of stock price and the Brownian motion prior to the time t_0 are known values ($\Im_{t_0}$-measurable). Without loss of generality, we also assume the realized values of stock price and the Brownian motion prior to the time t_0 are bounded.

Based on the risk-neutral pricing theory, the pricing of the ARMA-type contingent claims is done under the martingale probability measure Q by transforming the discounted stock price into a Q-martingale, which can be represented as

$$E_Q\left(S_t\middle|\Im_{t_0}\right) = S_{t_0}\exp(r(t-t_0)), \forall\, t \in [t_0, T]. \tag{16}$$

Applying the dynamics of the stock price in Eq. (12) and the definition of martingale measure Q, the transformation from probability measure P to martingale measure Q is provided in the following Lemma.

Lemma 3. *Assume that the dynamics of underlying stock price S satisfies Eq. (1). Let the roots of* $1-\alpha_1 z-\dots-\alpha_p z^p = 0$ *lie outside the unit circle. When the predictable process* φ *satisfies the following condition:*

$$A_n(t_0,t) + \sigma\sum_{i=0}^{n}\left(\psi_i\int_{t_0}^{t-ih}\varphi(s)ds\right) = 0, \forall\, t \in [t_n, t_{n+1}), \tag{17}$$

where n is the integer part of $(t-t_0)/h$ *, the process* W_t^Q *, which is given by the formula*

$$dW_t^Q = dW_t^P - \phi(t)dt, \tag{18}$$

follows a one-dimensional Brownian motion on the probability space $(\Omega, \Im, Q)$. *In addition, under martingale measure Q, the stock price process which satisfies Eq. (12) under probability measure P is of the form:*

$$\ln S_t = \ln S_{t_0} + r(t-t_0) - \frac{1}{2}V_n(t_0,t) + Z_N^Q(t_0,t), \forall t \in [t_n, t_{n+1}), \tag{19}$$

where n is the integer part of $(t-t_0)/h$ *and* $Z_N^Q(t_0,t) = \sigma\sum_{j=0}^{n}\psi_j\left(W_{t-jh}^Q - W_{t_0}^Q\right)$.

The proof of Lemma 3 is provided in Appendix C. Note that, given $A_n(t_0,t)$ in Eq. (13), the predictable process φ is chosen according to Eq. (17).

Therefore, by virtue of Eq. (3), it suggests that, under the physical probability measure P, the instantaneous stock returns following a stationary ARMA process can be transformed to an infinite-order MA process with new MA parameters ψ_j, $j=1, \dots, \infty$. Under the martingale measure Q, the stock return between time t_0 and time t only depends upon the MA parameters ψ_j, $j=1, \dots, n$, where n is the integer part of $(t-t_0)/h$.

3. ARMA option pricing model

The first aim of this section is to examine the European option on a specified stock whose return is modeled as an ARMA process. After recognizing the fact that the ARMA representation of instantaneous stock return defined in Eq. (1) allows no arbitrage opportunities, the closed-form solutions of ARMA(p, q)-type options are provided by using the martingale pricing method. Moreover, one interesting extension of ARMA-type options, volatility options, is briefly described.

3.1. ARMA European options: stock options

Taking plain vanilla European call and put options as examples, their payoffs at the expiry date T are correspondingly Max $(S_T - K,\ 0)$ and Max $(K - S_T,\ 0)$, where K is the strike price. The time-t_0 values of European call options C_{t_0} and European put options P_{t_0} are given by

$$C_{t_0} = e^{-r(T-t_0)} E_Q\left[Max(S_T - K, 0)\middle|\Im_{t_0}\right], P_{t_0} = e^{-r(T-t_0)} E_Q\left[Max(K - S_T, 0)\middle|\Im_{t_0}\right]. \tag{20}$$

The pricing formulas of the above are provided in the following Theorem.

Theorem 1. *When the stock price process S defined in Eq. (1) follows a stationary ARMA model, the closed-form solutions for the ARMA(p, q)-type European options are as follows:*

$$C_{t_0} = S_{t_0}\Phi(d_{1N}(t_0, T)) - Ke^{-r(T-t_0)}\Phi(d_{2N}(t_0, T)), \tag{21}$$

$$P_{t_0} = Ke^{-r(T-t_0)}\Phi\left(-d_{2N}\left(t_0, T\right)\right) - S_{t_0}\Phi\left(-d_{1N}\left(t_0, T\right)\right), \tag{22}$$

where

$$d_{1z}(t,s) = \frac{In\frac{S_t}{K} + \left(r + \frac{1}{2}\sigma_z^2(t,s)\right)(s-t)}{\sigma_z(t,s)\sqrt{s-t}}, d_{2z}(t,s) = d_{1z}(t,s) - \sigma_z(t,s)\sqrt{s-t} \tag{23}$$

$$\sigma_z^2(t,s) = \frac{V_z(t,s)}{s-t}, \tag{24}$$

z is the integer part of $(s-t)/h$ *for* $s \geq t$ *and* $\Phi(\cdot)$ *is the cumulative probability of a standard normal distribution.*

The proof of Theorem 1 is in Appendix D.

Eqs. (21) and (22) show that the closed-form solution for an ARMA(p, q)-type European option is the same as the Black–Scholes–Merton (BSM) formula, except that the volatility function $\sigma_z(t, s)$ depends upon the AR and MA parameters. It is noteworthy that the implied volatility estimated from the BS formula can be successfully interpreted as one calculated from an ARMA(p, q)-type option formula. Specifically, this finding demonstrates that the BSM implied volatility is also valid—even if the stock returns follow an ARMA process.

In the absence of the ARMA effect, Eqs. (21) and (22), indeed, reduce to the BSM pricing formulas. When the AR and MA parameters are zero except for $\beta_1 = \beta$, it can be verified that $V_N(t_0, T) = \sigma^2(T - t_0)1_{(N=0)} + 1_{(N>0)}\sigma^2(h + (\beta + 1)^2(T - t_1))$ for the first-order MA setup, and Eq. (21) reduces to the MA(1)-type call formula of Liao and Chen (2006). Furthermore, Eqs. (21) and (22) obviously indicate that the ARMA(p, q)-type option prices will eventually converge to the BSM prices when the time to maturity is approaching zero. This result is in agreement with the results of Roll (1977), Duan (1995), Heston and Nandi (2000) and Liao and Chen (2006), where the option value with one period to expiration obeys the BSM formula.

By recalling that $\psi_0 = 1$, Eq. (15') can be rewritten as follows:

$$V_n(t_0, t) = \sigma^2\left(1_{(n>0)}\sum_{j=0}^{n-1}\left(1 + \sum_{1=1}^{j}\psi_i\right)^2 h + \left(1 + \sum_{i=1}^{n}\psi_i\right)^2 (t - t_n)\right), \forall t \in [t_n, t_{n+1}), \tag{15''}$$

where n is the integer part of $(t - t_0)/h$. If the new MA parameters ψ_i for $i = 1,\dots, n$ are zero, $\sigma_n^2(t_0,\ t)$ satisfy

$$\sigma_n^2(t_0, t) = \frac{V_n(t_0, t)}{t - t_0} = \sigma^2\left(1_{(n>0)}\sum_{j=0}^{n-1} h + (t - t_n)\right)\Big/(t - t_0) = \sigma^2. \tag{25}$$

Thus, given the same initial stock price, exercise price, volatility, risk-free rate and time to maturity, Eq. (25) implies that the option values obtained from the BSM formula is the same as the one using the ARMA-type option formula when ψ_i for $i = 1,\dots, n$ is zero. Similarly, the BSM prices may undervalue (overvalue) the prices of ARMA-type options when ψ_i for $i = 1,\dots, n$ are all positive (negative).

Taking an ARMA(1,1)-type option as an example, Eq. (9) reduces to $\psi_0 = 1$ and $\psi_j = \alpha_1^{j-1}(\alpha_1 + \beta_1)$ for $j > 0$ because $c_1 = 1$ and $\lambda_1 = \alpha_1$. Therefore, the variance function $V_n(t_0, t)$ defined in Eq. (15") becomes

$$V_n(t_0, t) = \sigma^2\left(1_{(n>0)}\left(1 + \sum_{j=1}^{n-1}\left(1 + (\alpha_1 + \beta_1)\frac{1 - \alpha_1^j}{1 - \alpha_1}\right)^2\right)h + \left(1 + (\alpha_1 + \beta_1)\frac{1 - \alpha_1^n}{1 - \alpha_1}\right)^2 (t - t_n)\right), \forall t \in [t_n, t_{n+1}), \tag{26}$$

where n is the integer part of $(t-t_0)/h$. Two important issues concerning Eq. (26) are discussed. First, the parameters ψ_i for $i=1,\ldots,n$ are composed of α_1 and α_1, the BSM prices undervalue the prices of ARMA(1,1)-type options if $\alpha_1+\beta_1$ is non-negative, and vice versa. Second, it has been studied by Liao and Chen (2006) that the MA effect on option values is significant. In this paper, we demonstrate that the AR effect is more significant than the MA effect. Let $t=t_n$, Eq. (26) can be rewritten as

$$V_n(t_0,t_n)=\sigma^2 h\left(1+\sum_{j=1}^{n-1}\left(1+(\alpha_1+\beta_1)\frac{1-\alpha_1^j}{1-\alpha_1}\right)^2\right). \tag{26'}$$

For a MA(1) model, the variance function $V_n(t_0,t_n)$ becomes

$$V_n^{MA}(t_0,t_n)=\sigma^2 h\left(1+\sum_{j=1}^{n-1}(1+\beta_1)^2\right)=\sigma^2 h\Big(1+(1+\beta_1)^2(n-1)\Big), \tag{27}$$

which coincides with the variance function of Liao and Chen (2006). For a stationary AR(1) model with $|\alpha_1|<1$, the variance function $V_n(t_0,t_n)$ becomes

$$V_n^{AR}(t_0,t_n)=\sigma^2 h\left(1+\sum_{j=1}^{n-1}\left(1+\alpha_1\frac{1-\alpha_1^j}{\alpha_1}\right)^2\right)=\sigma^2 h\left(1+\sum_{j=1}^{n-1}\left(1+\alpha_1+\alpha_1^2+\ldots+\alpha_1^j\right)^2\right). \tag{28}$$

When $|\alpha_1|<1$ and the level of AR parameter α_1 is the same as that of MA parameter β_1, we have

$$\left(1+\alpha_1+\alpha_1^2+\ldots+\alpha_1^j\right)^2=\left(1+\alpha_1+\alpha_1^2\left(\frac{1-\alpha_1^{j-1}}{1-\alpha_1}\right)\right)^2\geq(1+\alpha_1)^{2.}=(1+\beta_1)^2, \tag{29}$$

which leads to $V_n^{AR}(t_0,t_n)\geq V_n^{MA}(t_0,t_n)$. Because the option values are determined by the variance function, with other things being the same, the option values obtained by AR(1) model is higher than those obtained by MA(1) model,which implies that the AR effect is more significant than the MA effect.

3.2. ARMA European options: volatility options

One application of ARMA pricing formula is for volatility options. The growing literature on volatility options has emerged after the 1987 crash. Brenner and Galai (1989, 1993) first suggested options written on a volatility index that would serve as the underlying asset. Toward this end, various pricing models for volatility options written on instantaneous volatility have also been developed. Whaley (1993) uses the Black (1976) futures model to price volatility options written on an implied volatility index; volatility futures options with a zero cost-of-carry is considered. Hence, under the martingale measure Q, he assumes implicitly that volatility is a traded asset that follows a geometric Brownian motion process. Grünbichler and Longstaff (1996) derive a closed-form expression to price European options on volatilities which follows a mean reverting square-root process. The underlying asset is the instantaneous volatility of the returns of a stock index. They regard volatility as a nontradable asset and that the market price of volatility risk has to be introduced. Along the line of the GL model, Detemple and Osakwe (2000), who assume that volatility follows a mean reverting logarithmic process but the market price of volatility risk is zero, provide an analytic pricing formula for European volatility options.

One alternative setup of the volatility process is an ARMA process. Precisely, many studies show that ARMA-based forecasts emerge as the superior model-based forecasts, including that of Alizadeh et al. (2002), Gallant et al. (1999), Ferland and Lalancette (2006), and Pong et al. (2004) just to name a few. Motivated by the above empirical findings, it is reasonable that the volatility follows a stationary ARMA process; that is

$$d\,In\sum\nolimits_t=\mu_{\sum}dt+\sum_{i=1}^{p}\alpha_i\,d\,In\sum\nolimits_{t-ih}+j=\sum_{j=0}^{q}\sigma\sum\beta_j\,dW_{t-jh,}^P \tag{30}$$

where Σ_t is the value of the volatility index at time t and Σ_t is the volatility of the volatility index returns. To make the volatility into a Q-martingale along the line of Lemma 3, the process defined in Eq. (22) can be given by

$$In\sum\nolimits_t=In\sum\nolimits_{t_0}+r(t-t_0)-\frac{1}{2}V_n^{\sum}(t_0,t)+\sigma_{\sum}\sum_{j=0}^{n}\psi_j\left(W_{t-jh}^Q-W_{t_0}^Q\right),\ \forall\, t\in[t_n,t_{n+1}), \tag{31}$$

where

$$V_n^{\sum}(t_0,t)=\sigma_{\sum}^2\left(1_{(n>0)}\sum_{j=0}^{n-1}\left(1+\sum_{i=1}^{j}\psi_i\right)^2 h+\left(1+\sum_{i=1}^{n}\psi_i\right)^2(t-t_n)\right),\ \forall\, t\in[t_n,t_{n+1}), \tag{32}$$

n is the integer part of $(t-t_0)/h$. Let CV_{t_0} and PV_{t_0} be, respectively, the time-t_0 values of European volatility call and put options and satisfy

$$CV_{t_0}=e^{-r(T-t_0)}E_Q[Max(\sum_T-K,0)|\Im_{t_0}],\ \ PV_{t_0}=e^{-r(T-t_0)}E_Q[Max(K-\sum_T,0)|\Im_{t_0})]. \tag{33}$$

Following the similar pricing procedure for Theorem 1 and the Whaley's assumption that views volatility options as volatility futures options with a zero cost-of-carry, the closed form solutions of European volatility options are given by the following Corollary.

Corollary 1. *Assume that the dynamics of the volatilities are given by Eq. (30), the ARMA(p, q)-type European volatility options are priced by the following:*

$$CV_{t_0}=e^{-r(T-t_0)}\left[\sum\nolimits_{t_0}\Phi(\ell_{1N}(t_0,T))-K\Phi(\ell_{2N}(t_0,T))\right], \tag{34}$$

$$PV_{t_0}=e^{-r(T-t_0)}\left[K\Phi(-\ell_{2N}(t_0,T))-\sum\nolimits_{t_0}\Phi(\ell_{1N}(t_0,T))\right], \tag{35}$$

where

$$\ell_{1z}(t,s)=\frac{In\frac{\sum_t}{K}+\frac{1}{2}v_z^2(t,s)(s-t)}{v_z(t,s)\sqrt{s-t}},\ell_{2z}(t,s)=\ell_{1z}(t,s)-v_z(t,s)\sqrt{s-t}, \tag{36}$$

$$v_z^2(t,s)=\frac{V_z^{\sum}(t,s)}{s-t}, \tag{37}$$

z is the integer part of $(s-t)/h$ *for* $s\geq t$.

Note that the closed-form solutions for ARMA-type European volatility options are the same as Whaley's formula, except that the volatility input $v_z(t,s)$ depends upon the AR and MA parameters. Using Monte Carlo simulation under a stochastic volatility setup, Psychoyios and Skiadopoulos (2006) find that the BSM and Whaley's models can be used reliably for pricing and hedging purposes. The option values calculated from the BSM and Whaley's formulas can be also regarded as the one priced by the ARMA option pricing models using the appropriate volatility input, which indicates that the ARMA option pricing model is a reliable alternative for pricing and hedging purposes.

4. Numerical analysis of ARMA-type options

To gauge the ARMA effect of stock return on the option's value, a one-month maturity European call option on stock is considered here, the instantaneous return of which follows an ARMA(1,1) process. The initial stock price and volatility are set to 100 and 30%, respectively. The risk-free interest rate is 5%. Moreover, the ARMA effect on option price may depend on the strike price of the European call option. Hence, regarding the strike price, three target options are considered: an in-the-money (ITM) call option, an at-the-money (ATM) call option, and an out-of-the-money (OTM) call option ($K=90$, 100 and 110, respectively).

Table 1 shows the ratio of one-month maturity ARMA(1,1)-type option prices to the BSM prices with different combinations of AR and MA parameters and various moneyness levels of the target option. Obviously, the option values increase as the level of AR or MA parameters increases across all moneyness levels. In particular, the AR effect is more significant than the MA effect. For example, the ratios of ARMA(1,1)-type option prices to the BSM prices which are respectively 69.71%, 81.78%,91.73%, 110.08%, 130.07% and 188.65% for $\alpha_1=-0.5$, -0.25, -0.1, 0.1, 0.25 and 0.5, when $\beta_1=0$ and strike price is 100, are larger than the ratios which are respectively 55.32%, 77.41%, 90.93%, 109.1%, 122.79% and 145.68% for $\beta_1=-0.5$, -0.25, -0.1, 0.1, 0.25 and 0.5, when $\alpha_1=0$ with the same strike price. In addition, in the case of $\alpha_1=0$, the absolute ratio increments for ITM, ATM and OTM options are respectively 12.09% (108.55%–96.46%), 90.36% (145.68%–55.32%) and 273.79% (279.46%–5.67%) as β_1 changes from -0.5 to 0.5. In the case of $\beta_1=0$, however, the absolute ratio increments for ITM, ATM and OTM options are respectively 22% (118.96%–96.96%), 118.94% (188.65%–69.71%), and 458.41% (481.51%–23.1%), as α_1 changes from -0.5 to 0.5. Thus, the AR effects dominate the MA effects across all moneyness levels.

In the case of $\alpha_1+\beta_1=0$ (in the main diagonal for each panel), the ratios of ARMA(1,1)-type option prices to the BSM prices are 100%, which means that the ARMA(1,1)-type option prices are equal to the BSM prices. Specifically, autocorrelation in stock return with $\alpha_1+\beta_1=0$ has no impact on option price; hence, given the same initial stock price, exercise price, risk-free rate, stock return volatility and time to maturity, using the BSM formula to price the ARMA-type option is also correct even when the stock price process follows an ARMA process with $\alpha_1+\beta_1=0$. However, in view of Eq. (26), if $\alpha_1+\beta_1>0$ ($\alpha_1+\beta_1<0$) resulting in $\sigma_n^2(t_0,t)>\sigma^2$ ($\sigma_n^2(t_0,t)<\sigma^2$), the BSM prices undervalue (overvalue) the ARMA(1,1)-type option prices. In particular, it is also observed that the impact of $\alpha_1+\beta_1$ is asymmetric across all moneyness levels. For example, in the case of the ITM option, the ARMA(1,1)-type option price is higher than the BSM price by 43.67% when $\alpha_1=\beta_1=0.5$ and is lower than the BSM price by

Table 1
One-month maturity option prices under ARMA(1,1)-type stock returns (daily frequency).

α_1								
BSM price	β_1	−0.5	−0.25	−0.1	0	0.1	0.25	0.5
Panel A. Strike price K = 90								
10.7661	0.50	100.00%	102.99%	105.90%	108.55%	112.02%	119.46%	143.67%
	0.25	98.13%	100.00%	101.95%	103.82%	106.33%	111.91%	130.97%
	0.10	97.34%	98.59%	100.00%	101.39%	103.33%	107.77%	123.65%
	0.00	96.96%	97.85%	98.91%	100.00%	101.55%	105.24%	118.96%
	−0.10	96.68%	97.26%	98.01%	98.81%	100.00%	102.93%	114.46%
	−0.25	96.45%	96.69%	97.04%	97.47%	98.15%	100.00%	108.22%
	−0.50	96.36%	96.37%	96.41%	96.46%	96.57%	96.99%	100.00%
Panel B. Strike price K = 100								
3.6349	0.50	100.00%	118.35%	133.30%	145.68%	160.74%	190.58%	277.71%
	0.25	84.75%	100.00%	112.46%	122.79%	135.37%	160.30%	233.19%
	0.10	75.69%	89.05%	100.00%	109.10%	120.18%	142.15%	206.47%
	0.00	69.71%	81.78%	91.73%	100.00%	110.08%	130.07%	188.65%
	−0.10	63.79%	74.56%	83.49%	90.93%	100.00%	118.02%	170.85%
	−0.25	55.07%	63.85%	71.23%	77.41%	84.97%	100.00%	144.18%
	−0.50	41.35%	46.56%	51.28%	55.32%	60.31%	70.32%	100.00%
Panel C. Strike price K = 110								
0.6681	0.25	55.32%	100.00%	143.36%	182.98%	234.69%	345.73%	707.12%
	0.10	34.31%	66.82%	100.00%	131.17%	172.69%	263.92%	570.31%
	0.00	23.10%	47.93%	74.45%	100.00%	134.67%	212.50%	481.51%
	−0.10	14.28%	32.02%	52.12%	72.15%	100.00%	164.32%	272.83%
	−0.25	5.49%	14.34%	25.73%	37.93%	55.87%	100.00%	272.83%
	−0.50	0.36%	1.32%	3.15%	5.67%	10.17%	24.14%	100.00%

Table 1 reports the ratio of one-month maturity ARMA(1,1)-type option prices to the BSM prices with different combinations of AR and MA parameters and various moneyness levels of the target option. The initial stock price and volatility are set to 100 and 30%, respectively. The risk-free interest rate is 5%.

3.64% as $\alpha_1 = \beta_1 = -0.5$. Accordingly, in the case of lower values of the AR and MA parameters and the smaller strike prices, it is obvious that the difference between the ARMA(1,1)-type call prices and BSM prices is insignificant.

Given the AR or MA parameters, the absolute percentage differences between the prices and the ARMA (1,1)-type option prices will depend on the moneyness level. The absolute difference is minimized for ITM options and it is maximized for the OTM options. For example, the absolute percentage differences, which are correspondingly 3.64% (100%–96.36%), 58.65% (100%–41.35%) and 99.64% (100%–0.36%) for ITM, ATM and OTM options in the case of $\alpha_1 = \beta_1 = -0.5$, are likewise 1.4367, 2.7771 and 9.4161 times the BSM price for the ITM,ATM and OTM options when $\alpha_1 = \beta_1 = 0.5$, which indicates that the ARMA effects on option values depend on not only the level of AR and MA parameters but also the moneyness of the target option.

5. Empirical analysis of ARMA-type options

In this section, we use the TAIEX option contracts (calls and puts), ranked 3rd globally after KOSPI 200 options and Dow Jones Euro Stoxx 50 option contracts, to empirically test the ARMA-type option pricing model. TAIEX options are the most active products with a turnover of 92,757,254 contracts traded in 2008 accounting for 67.84% of total Taiwan futures and option market volume.

The TAIEX options have several features: (1) the TAIEX options are European-style; (2) both the TAIEX options and futures are cash settled; (3) the expiration days of the TAIEX futures coincide with those of the TAIEX options; (4) the TAIEX options and futures are traded side by side on the same exchange involving the same clearing house. Therefore, the TAIEX options can be priced as if they are European-style futures options with both the option and futures sharing the same maturity. The last feature allows us to bypass the difficult task of determining the appropriate dividend yield for the TAIEX. In the following section, we empirically test the ARMA-type option pricing model.[2]

5.1. Data

The empirical test is performed with daily closing prices of TAIEX futures and option data from 2004 to 2008 obtained directly from the Taiwan Futures Exchange. We construct the raw daily data set by using the closing option prices for each trading day. For each raw daily data set, the expiration months of TAIEX options include spot month, the next two calendar months and the next two quarterly months. In addition, there are at least five in-the-money option series and five out-of-the-money option series for

[2] Similar to Black (1976) for pricing futures options, the ARMA-type futures option pricing formulas can be obtained by substituting asset price S_{t_0} with $Fe^{r(T-t_0)}$ in Eqs. (21) and (22), where F is the futures price at time t_0 for settlement date T.

the spot month and the next two calendar months and three in-the-money option series and three out-of-the-money option series for the next two quarterly months.

To avoid liquidity-related biases, some filtering rules are applied to the raw daily data sets. First, option prices that are less than 1 are not used to mitigate the impact of price discreteness (the minimum tick for TAIEX option is 0.1). Lim and Guo (2000) use similar rule for S&P 500 futures option data. In terms of maturity, similar to Dumas et al. (1998), options with time to maturity less than 5 days or greater than half year are excluded. Third, a transaction must satisfy the no-arbitrage boundary of put-call parity in which the absolute value of call price plus the present value of strike price minus put price and the present value of futures price should be less than or equal to ten points reflecting the transaction costs and profit margin.[3] Finally, since the maximum estimated parameters is up to eight, daily data sets with option data less than ten are excluded. Therefore, from January 2, 2004 to December 31, 2008, there are 973 filtered daily data sets (filtered trading days) with 19,930 observations.

To construct the in-sample daily data sets, we first estimate the parameters for each model by minimizing the sum of squared errors between model prices and market prices for each filtered daily data set, allowing the parameters to change for each data set. Then, we use the in-sample parameters to obtain out-of-sample model prices for the adjacent data set. For example, we estimate the parameters for all competing models on January 2, 2004 (the first daily data set), then using the in-sample parameters to obtain the out-of-sample model prices on January 5, 2004 (the second daily data set). Therefore, there are 972 in-sample data sets and 972 out-of-sample data sets.

5.2. Parameter estimation

In this subsection, using the TAIEX option data from 2004 to 2008, we empirically compare the ARMA-type models with BSM model, ad hoc BSM model, variance gamma model and GARCH(1,1) model. According to the BSM model, the underlying asset follows a geometric Brownian motion and the daily asset returns are uncorrelated. Since empirical evidence demonstrates the predictability of asset returns, we provide a continuous-time ARMA model to capture the serial correlations in asset returns. However, when stock price process follows an ARMA process, the closed-form solution for an ARMA-typed European option is virtually the same as the BSM formula, except that the volatility function depends on the AR and MA parameters. As a result, similar to the BSM formula, the ARMA-type model cannot successfully explain the volatility skew phenomenon.

Much effort has been put into improving the BSM model to explain the volatility smile/skew phenomenon. To model and obtain a smooth implied volatility surface, Dumas et al. (1998) provide an ad hoc BSM model with the flexibility of fitting to the strike and term structure of observed implied volatilities. Specifically, volatility function is of the form

$$\sigma_{BSM} = a_0 + a_1 X + a_2 X^2 + a_3 \tau + a_4 \tau^2 + a_5 X \tau, \tag{38}$$

where σ_{BSM} is the BSM implied volatility for an option with strike price X and time to maturity τ. Incorporating the concept of an ad hoc BSM model into the ARMA(1,1) model, we provide an ad hoc ARMA(1,1) model by modifying the total volatility function in Eq. (26) as follows:

$$V_n(t_0,t) = \sigma_{ARMA}^2 \left(1_{(n>0)} \left(1 + \sum_{j=1}^{n-1} \left(1 + (\alpha_1+\beta_1) \frac{1-\alpha_1^j}{1-\alpha_1} \right)^2 \right) h + \left(1 + (\alpha_1+\beta_1) \frac{1-\alpha_1^n}{1-\alpha_1} \right)(t-t_n) \right), \forall t \in [t_n, t_{n+1}), \tag{39}$$

where n is the integer part of $(t-t_0)/h$ and σ_{ARMA}^2 is of the form:

$$\sigma_{ARMA} = b_0 + b_1 X + b_2 X^2 + b_3 \tau + b_4 \tau^2 + b_5 X \tau. \tag{40}$$

Note that the ad hoc ARMA(1,1) model reduces to ad hoc BSM model when the sum of AR and MA coefficients is zero and reduces to ARMA(1,1) model when all b_i coefficients are zero. Therefore, the ad hoc ARMA(1,1) model is a generalized option pricing model and is able to explain two properties: the volatility skew phenomenon and the predictable risk-neutral asset returns.

The presence of a volatility smile suggests that (risk neutral) asset return distributions are substantially skewed and leptokurtic in contrast with the normal asset returns in the BSM framework. Therefore, a second line of research is based on the postulate that the underlying asset follows a pure jump process. One of the most popular Lévy process in Finance is the variance gamma (VG) process introduced by Madan and Seneta (1990) for stock returns and Madan et al. (1998) and Hirsa and Madan (2004) for option pricing. The VG process is a Brownian motion on a stochastic business time that follows a stationary gamma process. The VG process permits more flexibility in modeling skewness and kurtosis relative to Brownian motion; therefore it is suited for the modeling of stock return distributions. In equity and interest rate modeling, the VG process has already proven its modeling capabilities because it gives a

[3] In Taiwan, the transaction cost per TAIEX option is around 2 points. Arbitrage transaction can be easily executed by using a box spread with two calls and two puts if the put-call parity condition is violated. As a result, the transaction cost is around $4 times 2 = 8$ points. In addition, we assume that the profit margin for each transaction should be greater than 2 points. In sum, the critical point for no-arbitrage price boundary is ten index points.

better fit to market option prices than the BSM model (Avramidis & L'Ecuyer, 2006; Schoutens, 2003). One approach to option pricing under VG process is based on the characteristic function representations (Carr & Madan, 1999) as follows:

$$C_{VG}(X,\tau) = \frac{e^{-y\,\ln X}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\,\ln X}\frac{e^{-r\tau}\phi\tau(\omega-(y+1)i)}{y^2+y-\omega^2+i(2y+1)\omega}d\omega, \tag{41}$$

where $i=\sqrt{-1}$, X is strike price, τ is time to maturity, y is chosen according to $\phi(-(y+1)i),<\infty$[4] and $\phi_\tau(\omega)$, the characteristic function of log stock price under martingale measure Q, is defined as

$$\phi_\tau(\omega) = \exp\left\{i\omega\left[\ln S_0+\left(r+\frac{1}{K}\ln\left(1-\theta K-\sigma_{VG}^2K/2\right)\right)\tau\right]\right\}\left(1+\frac{\sigma_{VG}^2\omega^2K}{2}-i\theta K\omega\right)^{-\frac{\tau}{K}}, \tag{42}$$

where σ_{VG} is a volatility parameter, θ is a skewness parameter and κ is a kurtosis parameter. Therefore, the VG model is a three-parameter model controlling over the skewness and kurtosis of the risk-neutral return distribution. An additional attractive feature of the VG model is that it nests the lognormal density and the BSM formula as a parametric special case.

A third line of research to capture the volatility clustering, leverage effect and volatility smile phenomenon is to model the dynamics of asset returns by using the discrete-time generalized autoregressive conditional heteroskedasticity (GARCH). Duan (1995) has developed an option pricing model in which the underlying asset follows a symmetric GARCH process of Bollerslev (1986). However, the option values in Duan model are solved by simulation that can be slow and computationally intensive for empirical work. Heston and Nandi (2000) develop a closed-form option formula for a spot asset whose variance follows an asymmetric GARCH(1,1) process with leverage effect under martingale measure Q as follows:

$$h_S(t+1) = \omega_S + a_S h_S(t) + b_S\left(z_S^Q(t)-(y_S+\lambda_S)\sqrt{h_S(t)}\right)^2, \tag{43}$$

where $h_S(t+1)$ is the conditional variances function at time $t+1$, $\omega_S>0$, $a_S\le 0$ and $b_S\le 0$. b_S determines the kurtosis of the distribution and b_S being zero implies a deterministic time varying variance. λ_S is the risk premium parameter. γ_S controls the skewness or the asymmetry of the distribution of the log returns. $z_S^Q(t)$ is a standard normal risk-neutral distribution.

According to the empirical analysis on S&P500 index options, Heston and Nandi (2000) find that the out-of-sample valuation errors from the GARCH(1,1) model are substantially lower than the ad hoc BSM model of Dumas et al. (1998). Therefore, in this paper, we also use the closed-form solution of Heston and Nandi (2000) as the competing model in the empirical analysis. Similarly, the GARCH(1,1) option pricing formula can be obtained by using the characteristic function representations as follows:

$$C_{GARCH}(X,\tau) = \frac{e^{-y\,\ln X}}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\,\ln X}\frac{e^{-r\tau}F\tau(i\omega+(y+1))}{y^2+y-\omega^2+i(2y+1)\omega}d\omega, \tag{44}$$

where the generating function $F(\phi)$ takes the following log-linear form:

$$F(\phi) = E_Q\left(S_\tau^\phi\right) = S_\tau^\phi e^{\phi rN} f(0,M,\phi), \tag{45}$$

where M is the integer part of τ/h, $f(0,M,\phi)$ is obtained by recursion as follows:

$$f(i,M,\phi) = \exp[B_S(i,M,\phi)+C_S(i,M,\phi)h_S(t+i+1)], \tag{46}$$

$$B_S(i-1,M,\phi) = B_S(i,M,\phi)+\omega_S C_S(i,M,\phi)-\frac{1}{2}\ln(1-2b_S C_S(i,M,\phi)), \tag{47}$$

$$C_J(i-1,M,\phi) = a_S C_S(i,M,\phi)+\phi\gamma_S^Q-\frac{1}{2}\left(\phi+\left(\gamma_S^Q\right)^2\right)+\frac{\left(\gamma_S^Q-\phi\right)^2}{2(1-2b_S C_S(i,M,\phi))}, \tag{48}$$

[4] As long as y satisfies $\phi(-(y+1)i)=E_Q(S_\tau^{y+1})<\infty$,the choice of decay rate parameter y is of no importance for European option pricing model. Matsuda (2004) recommended the choice of $1\le y\le 2$ for the VG option pricing model. As a result, we choose $y=1.25$ in this paper.

where $\gamma_S^Q = \gamma_S + \lambda_S$. The initial conditions are given by

$$B_S(M-1, M, \phi) = 0, C_S(M-1, M, \phi) = \frac{\phi}{2}(\phi - 1). \quad (49)$$

Therefore, the pricing formula of the GARCH(1,1) model depends on the five parameters: ω_s, a_S, b_S, λ_S and γ_S.

For model comparison, ad hoc BSM model, VG model and GARCH(1,1) model are employed as the competing models in the empirical analysis. Note that ad hoc BSM model, ad hoc ARMA(1,1) model, variance gamma model and GARCH(1,1) model can explain the volatility smile/skew phenomenon. In addition, when the ARMA-type models provide a smaller pricing error for both in-sample and out-of-sample fits, the risk-neutral asset returns could be predictable. When the VG model provides better in-sample and out-of-sample fits, the risk neutral asset returns could be independent (non Gaussian) VG distributed with skewness and excess kurtosis. When the GARCH(1,1) model provides better in-sample and out-of-sample fits, it is important to consider both the stochastic nature of volatility and correlation between volatility and asset returns. If ad hoc ARMA(1,1) model provides better in-sample and out-of-sample fits, the risk-neutral return distribution could be predictable, and the volatility function may be related to strike price and maturity.

For in-sample fit, we estimate the parameters for each model by minimizing the sum of squared errors between model prices and market prices for each data set, allowing the parameters to change. For out-of-sample comparison, we use the in-sample parameters to obtain each out-of-sample model price (OP_E) and calculate the ratio of the mean absolute error to the average option price (AVP) as follows:

$$MAE = \frac{1}{AVP} \frac{\sum_{i=1}^{N_S} |(OP_E - OP_M)|}{N_S}, \quad (50)$$

where N_S is the total numbers of out-of-sample option data and OP_M is the corresponding market option price. Finally, the proxy for risk-free rate is three-month commercial paper rate, the most active traded short-term rate in Taiwan financial market.

5.3. Model comparisons

This section evaluates in-sample and out-of-sample performance of the BSM model, ad hoc BSM model, ARMA(1,1) model, ad hoc ARMA(1,1) model, variance gamma model and GARCH(1,1) model. Table 2 reports in-sample comparisons across models examined. The average option price is 232.13 for overall in-sample option data. The aggregate MAE ratios are 2.91%, 2.84%, 2.09%, 2.81% and 2.90% for ad hoc BSM model, ARMA(1,1) model, ad hoc ARMA(1,1) model, variance gamma model and GARCH(1,1) model respectively, indicating that ARMA-type models or options pricing models taking into account the volatility smile/skew phenomenon can provide better in-sample fit than does the BSM model.

For the years 2004 and 2006, the ARMA(1,1) model outperforms other non-ARMA-type models, which implies that the risk neutral asset returns may tend to be predictable for the three years. For years 2007 and 2008, however, the models considering the volatility smile/skew phenomenon provide smaller valuation errors than does the ARMA(1,1) model. As a result, it is important to incorporate the ARMA(1,1) model with the models which can explain the volatility smile/skew phenomenon. The ad hoc ARMA(1,1) model falls in this category, attempting to capture the predictability property (ARMA effect) and to infer the volatility function from the option prices across different strikes and maturities (ad hoc effect). From Table 2, when we incorporate the ARMA(1,1) model with ad hoc effect for year 2007 (2008), the in-sample pricing errors reduce from 3.37% (3.24%) to 2.20% (1.85%). In addition, for

Table 2
In-sample model comparison.

	Models						Average premium	Number of observations
	BSM	Ad hoc BSM	ARMA	Ad hoc ARMA	VG	GARCH		
Parameters	1	6	3	8	3	5		
Panel A: aggregate pricing errors all years								
	3.85%	2.91%	2.84%	2.09%	20.81%	2.90%	232.13	19,908
Panel B: pricing errors per years								
2004	3.24%	2.57%	2.28%	2.06%	2.71%	2.89%	221.85	5376
2005	3.47%	2.96%	2.64%	2.39%	2.87%	3.07%	163.73	4392
2006	3.79%	3.12%	2.68%	2.18%	2.72%	2.90%	197.84	3342
2007	4.51%	3.27%	3.37%	2.20%	2.76%	2.93%	268.46	2662
2008	4.28%	2.88%	3.24%	1.85%	2.95%	2.80%	322.44	4136

Panel A reports the aggregate in-sample MAE (in percentage terms) from daily estimation by minimizing the sum of squared errors between model prices and market prices for BSM, ad hoc BSM, ARMA(1,1), ad hoc ARMA(1,1), variance gamma and GARCH models. Panel B reports the in-sample MAE (in percentage terms) for each year. MAE is the ratio of the mean absolute error to the average option price. Average premium is the average option price for the overall in-sample option data.

the overall in-sample data, the ad hoc ARMA(1,1) model also provide the smallest in-sample pricing errors. Therefore, based on the in-sample fit, the empirical evidence supports the ad hoc ARMA(1,1) model for pricing TAIEX options.

Table 3 reports the out-of-sample pricing errors for various models. From Table 3, the aggregate MAE ratios are respectively 4.80%, 4.62%, 4.42%, 4.40% and 4.49% for ad hoc BSM model, ARMA(1,1) model, ad hoc ARMA(1,1) model, variance gamma model and GARCH(1,1) model. Therefore, for both in-sample and out-of-sample fits, the ARMA-type models or the pricing models explaining the volatility smile/skew phenomenon can provide smaller aggregate pricing errors than the BSM model.

In view of the smallest out-of-sample pricing errors, the variance gamma model outperforms other models and provides the best price prediction for TAIEX options, which means that the risk neutral returns may be substantially skewed and leptokurtic. In addition, the estimated parameters of ARMA(1,1) model, variance gamma model, GARCH(1,1) model, ad hoc BSM model and ad hoc ARMA(1,1) model are correspondingly 3, 3, 5, 6 and 8. As shown in Tables 2 and 3, the variance gamma model with three parameters has the second best in-sample fit and the best out-sample fit. Therefore, for parameter parsimony purpose, the variance gamma model is a superior model for pricing TAIEX options. However, the pricing errors of ad hoc ARMA(1,1) model (4.42%) are as small in aggregate MAE ratio as that of variance gamma model (4.4%). Consequently, the ad hoc ARMA(1,1) model with the best in-sample fit is also a good candidate for pricing TAIEX options.

6. Conclusions

The evidence shows that daily, weekly and monthly returns are predictable from past returns. Motivated by the empirical findings that asset returns or volatilities are predictable, this paper studies the pricing of European options on the stock or volatility, the instantaneous logarithm increments of which depend upon an ARMA process. The pricing formula of an ARMA-type option is similar to BSM formula, except for the total volatility input depending upon the AR and MA parameters. Consequently, the implied volatility estimated from the BSM formula can be successfully interpreted as the one calculated from an ARMA-type option formula. Specifically, this finding demonstrates that the BSM implied volatility is also valid even if the instantaneous stock returns follow an ARMA process.

In the absence of ARMA effects, the ARMA option pricing formula, indeed, reduces to the BSM pricing formula. When the AR and MA parameters are equal to zero (except for the first order MA coefficient), the ARMA option pricing formula reduces to the MA(1)-type option formula of Liao and Chen (2006). Furthermore, the ARMA-type option prices eventually converge to the BSM price when the time-to-maturity is approaching zero. This result is in agreement with the assumption of Roll (1977), Duan (1995), Heston and Nandi (2000) and Liao and Chen (2006), where the option value with one period to expiration obeys the BSM formula. Based on the result of numerical analyses, the option values are increasing functions of the level of AR or MA parameters for all moneyness levels. Specifically, the AR effect is more significant than the MA effect.

Using the TAIEX option data from 2004 to 2008, we empirically compare the ARMA-type models with BSM model, ad hoc BSM model, variance gamma model and GARCH(1,1) model. From the empirical results, the ad hoc ARMA(1,1) model provides the best in-sample fit, whereas the variance gamma model provides the best out-of-sample fit. As a result, both the ad hoc ARMA(1,1) model and the variance gamma model are excellent pricing models for TAIEX options. Furthermore, from the empirical study, since both the ad hoc ARMA(1,1) model and the variance gamma model can provide smaller valuation errors, it is an interesting topic by extending the ARMA model to the ARMA-Lévy models by incorporating the ARMA process with the variance gamma process or other Lévy processes such as normal inverse Gaussian process. Since the ARMA-Lévy models can also be reduced to an ARMA model or Lévy model, it accounts for the negative skewness and positive excess kurtosis for the predictable returns under risk neutral measure, and it is expected to provide smaller valuation errors for pricing TAIEX options.

Table 3
Out-of-sample pricing errors.

	Models						Average premium	Number of observations
	BSM	Ad hoc BSM	ARMA	Ad hoc ARMA	VG	GARCH		
Parameters	1	6	3	8	3	5		
Panel A: aggregate pricing errors across all years								
	5.12%	4.80%	4.62%	4.42%	4.40%	4.49%	232.25	19,896
Panel B: pricing errors per years								
2004	4.45%	4.52%	4.01%	4.16%	4.20%	4.34%	222.19	5342
2005	4.13%	4.11%	3.54%	3.84%	3.59%	3.85%	163.73	4392
2006	4.82%	4.58%	4.00%	4.31%	3.90%	4.00%	197.84	3342
2007	6.41%	5.76%	5.84%	5.30%	4.95%	5.25%	268.46	2662
2008	5.72%	5.00%	5.38%	4.53%	4.96%	4.80%	322.03	4158

Panel A reports the aggregate out-of-sample MAE (in percentage terms) for BSM, ad hoc BSM, ARMA(1,1), ad hoc ARMA(1,1), variance gamma and GARCH models. Panel B reports the MAE (in percentage terms) for each year. MAE is the ratio of the mean absolute error to the average option price. Average premium is the average option price for the overall out-of-sample option data.

Appendices

Appendix A. The proof of Lemma 1

To ensure that the instantaneous asset returns follow a stationary ARMA model, we assume that the roots of $1-\alpha_1 z-\ldots-\alpha_p z^p=0$ lie outside the unit circle, which implies that $\sum_{j=0}^{\infty}\left|\psi_j\right|<\infty$ (Hamilton, 1994). Therefore, both sides of Eq. (2) can be divided by $1-\alpha_1 z-\ldots-\alpha_p z^p=0$ to obtain Eq. (3). According to Definition 2.4 of Karatzas and Shreve (1991) and Definition 5.1 of Klebaner (1998), $\ln S_t$ is called a strong solution of the stochastic differential equation (SDE) defined in Eq. (3) with initial condition S_{t_0} if integral $\sum_{j=0}^{\infty}\int_{t_0}^{t}\psi_j dW_{u-jh}^P=\sum_{j=0}^{\infty}\int_{t_0}^{t-jh}\psi_j dW_u^P$ exists. The integral version of Eq. (3) is of the form:

$$InS_t=InS_{t_0}+\mu(t-t_0)+\sigma\sum_{j=0}^{\infty}\int_{t_{0-jh}}^{t-jh}\psi_j dW_u^P. \tag{5}$$

Note that the requirement on the existence of the integral $\sum_{j=0}^{\infty}\int_{t_{0-jh}}^{t-jh}\psi_j dW_u^P$ is that its variance satisfies $V\left(\sum_{j=0}^{\infty}\int_{t_0-jh}^{t-jh}\psi_j dW_u^P\right)<\infty$. Because $V\left(\int_{t_0-jh}^{t-jh}\psi_j dW_u^P\right)=V\left(\int_{t_0}^{t}\psi_j dW_u^P\right)=\psi_j^2(t-t_0)$ and $Cov(W_{t-jh}^P-W_{t-jh}^P, W_{t_{0-ih}}{}^P-W_{t_{0-ih}}{}^P)=t-t_0-\max(i-j,j-i)h\le V(W_t^P-W_{t_0}{}^P)=t-t_0$, we have

$$\begin{aligned}V\left(\sum_{j=0}^{\infty}\int_{t_{0-jh}}^{t-jh}\psi_j dW_u^P\right)&\le V\left(\sum_{j=0}^{\infty}\int_{t_0}^{t}\psi_j dW_u^P\right)=V\left(\left(\sum_{j=0}^{\infty}\psi_j\right)\left(W_t^P-W_{t_0}^P\right)\right)\\&=\left(\sum_{j=0}^{\infty}\psi_j\right)^2 V\left(W_t^P-W_{t_0}^P\right)\le\left(\sum_{j=0}^{\infty}\left|\psi_j\right|\right)^2(t-t_0)<\infty.\end{aligned} \tag{A.1}$$

Consequently, $\ln S_t$ defined in Eq. (5) is a strong solution if the roots of lie outside the unit circle. In addition, the strong solution is unique because the drift and diffusion coefficients in Eq. (5) are constant which automatically satisfies the requirements of the uniqueness of strong solutions that the drift and diffusion coefficients satisfy the Lipschitz and linear growth conditions defined in Theorem 2.9 of Karatzas and Shreve (1991) or Theorem 5.2 of Klebaner (1998). This completes the proof of the Lemma.

Appendix B. The proof of Lemma 2

As proved by Proposition 2.2 of Hamilton (1994), factoring the pth-order polynomial in the lag operator $1-\alpha_1 L-\ldots-\alpha_p L^p=(1-\lambda_1 L)(1-\lambda_2 L)\ldots(1-\lambda_p L)$ is the same calculation as finding the eigenvalues of the matrix F defined in Eq. (8). In addition, assuming that the roots of $1-\alpha_1 z-\ldots-\alpha_p z^p=0$ lie outside the unit circle implies that the eigenvalues $(\lambda_1, \lambda_2, \ldots,\lambda_p)$ lie inside the unit circle. Therefore, Eq. (1) can be rewritten as follows:

$$dInS_t=\mu dt+\sigma\left(1+\beta_1 L+\ldots+\beta_q L^q\right)\left((1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}\ldots\left(1-\lambda_p L\right)^{-1}\right)dW_t^P, \tag{B.1}$$

Sargent (1987, pp. 192–193) proves that

$$(1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}\ldots\left(1-\lambda_p L\right)^{-1}=\sum_{i=1}^{p}\frac{c_i}{1-\lambda_i L}=\sum_{y=0}^{\infty}\left(\sum_{i=1}^{p}c_i(\lambda_i L)^y\right), \tag{B.2}$$

where c_i is defined in Eq. (10) and satisfies $\sum_{i=1}^{p}c_i=1$. Substituting Eq. (B.2) into Eq. (B.1) yields

$$dInS_t=\mu dt+\mu dt+\sigma\sum_{j=0}^{q}\left\{\left(\sum_{y=0}^{\infty}\left(\sum_{i=1}^{p}c_i(\lambda_i L)^y\right)\right)\beta_j dW_{t-j}^P\right\}. \tag{B.3}$$

Expanding the last term in Eq. (B.3) and sorting them according to dW^P_{t-j}, we can obtain Eq. (3). For example, when $q=2$, Eq. (B.3) is given by

$$
\begin{aligned}
d\,In\,S_t &= \mu dt + \sigma\beta 0\left[\left(\sum_{i=1}^{p} c_i\right)dW^P_t + \left(\sum_{i=1}^{p} c_i\lambda_i\right)dW^P_{t-1} + \left(\sum_{i=1}^{p} c_i\lambda_i^2\right)dW^P_{t-2} + \left(\sum_{i=1}^{p} c_i\lambda_i^3\right)dW^P_{t-3} + \ldots\right] \\
&\quad + \sigma\beta_1\left[\left(\sum_{i=1}^{p} c_i\right)dW^P_{t-1} + \left(\sum_{i=1}^{p} c_i\lambda_i\right)dW^P_{t-2} + \left(\sum_{i=1}^{p} c_i\lambda_1^2\right)dW^P_{t-3} + \left(\sum_{i=1}^{p} c_i\lambda_i^3\right)dW^P_{t-4} + \ldots\right] \\
&\quad + \sigma\beta_2\left[\left(\sum_{i=1}^{p} c_i\right)dW^P_{t-2} + \left(\sum_{i=1}^{p} c_i\lambda_i\right)dW^P_{t-3} + \left(\sum_{i=1}^{p} c_i\lambda_i^2\right)dW^P_{t-4} + \left(\sum_{i=1}^{p} c_i\lambda_i^3\right)dW^P_{t-5} + \ldots\right] \\
&= \mu dt + \sigma\left(\sum_{i=1}^{p} c_i\beta_0\right)dW^P_t + \sigma\left(\sum_{i=1}^{p} c_i(\beta_1+\lambda_i)\right)dW^P_{t-1} + \sigma\left(\sum_{i=1}^{p} c_i\left(\beta_2+\beta_1\lambda_i+\lambda_i^2\right)\right)dW^P_{t-2} \\
&\quad + \sigma\left(\sum_{i=1}^{p} c_i\lambda_i\left(\beta_2+\beta_1\lambda_i+\lambda_i^2\right)\right)dW^P_{t-3} + \sigma\left(\sum_{i=1}^{p} c_i\lambda_i^2\left(\beta_2+\beta_1\lambda_i+\lambda_1^2\right)\right)dW^P_{t-4} + \ldots
\end{aligned}
$$

Therefore, $\psi_0 = \beta_0\sum_{i=1}^{p} c_i = 1$, and $\psi_1 = \sum_{i=1}^{p} c_i(\beta_1+\beta_1+\lambda_i)$, which is consistent with Eq. (9). This completes the proof of the Lemma.

Appendix C. The proof of Lemma 3

Substituting Eq. (18) into Eq. (12) yields

$$
\begin{aligned}
In\,S_t &= In\,S_{t_0} + r(t-t_0) - \frac{1}{2}V_n(t_0,t) + A_n(t_0,t)\sigma\sum_{j=0}^{n}\left(\psi_j\int_{t_0}^{t-ih} dW^P_u\right) \\
&= In\,S_{t_0} + r(t-t_0) - \frac{1}{2}V_n(t_0,t) + A_n(t_0,t)\sigma\sum_{j=0}^{n}\left(\psi_j\left(\int_{t_0}^{t-ih} dW^Q_u + \int_{t_0}^{t-ih}\varphi(u)du\right)\right) \\
&= In\,S_{t_0} + r(t-t_0) - \frac{1}{2}V_n(t_0,t) + \left(A_n(t_0,t)\sigma\sum_{i=0}^{n}\left(\psi_i\int_{t_0}^{t-ih}\varphi(u)du\right)\right) + \sigma\sum_{i=0}^{n}\left(\psi_j\int_{t_0}^{t-ih} dW^Q_u\right) \\
&= In\,S_{t_0} + r(t-t_0) - \frac{1}{2}V_n(t_0,t) + Z^Q_n(t_0,t) + \left(A_n(t_0,t)\sigma\sum_{i=0}^{n}\left(\psi_i\int_{t_0}^{t-ih}\varphi(u)du\right)\right).
\end{aligned}
\quad \text{C.1}
$$

Due to the fact that $E_Q\left(\exp\left(Z^Q_n(t_0,t)\right)\middle|\Im_{t_0}\right) = \exp(\frac{1}{2}V_n(t_0,t))$, we have

$$
\begin{aligned}
E_Q\left(S_t\middle|\Im_{t_0}\right) &= S_{t_0} e^{\left(r(t-t_0) - \frac{1}{2}V_n(t_0,t) + \sigma\sum_{i=0}^{n}\left(\psi_i\int_{t_0}^{t-ih}\varphi(u)du\right)\right)} E_Q\left(e^{Z^Q_n(t_0,t)}\middle|\Im_{t_0}\right) \\
&= S_{t_0}\exp\left(r(t-t_0) + A_n(t_0,t) + \sigma\sum_{i-0}^{n}\left(\psi_i\int_{t_0}^{t-ih}\varphi(u)du\right)\right).
\end{aligned}
\quad \text{C.2}
$$

Therefore, we arrive at Eq. (16) if and only if

$$
A_n(t_0,t) + \sigma\sum_{i=0}^{n}\left(\psi_i\int_{t_0}^{t-ih}\varphi(s)ds\right) = 0, \forall t\in\left[t_n, t_{n+1}\right). \quad \text{(C.3)}
$$

where n is the integer part of $(t-t_0)/h$. However, to apply the martingale pricing method, the requirement that no arbitrage exists needs to check the prerequisite condition of Girsanov's theorem that

$$
\int_{t_0}^{t}(\varphi(s))^2 ds < \infty. \quad \text{(C.4)}
$$

By virtue of Eq. (C.3), Eq. (C.4) can be proved by using mathematical induction. First, for the case of $n=0$, we have

$$
A_0(t_0,t) + \sigma\sum_{i=0}^{0}\left(\psi_i\int_{t_0}^{t-ih}\varphi(s)ds = A_0(t_0,t) + \sigma\int_{t_0}^{t}\varphi(s)ds\right) = 0. \quad \text{(C.5)}
$$

According to Eqs. (12) and (15'), we have

$$\begin{aligned} A_0(t_0,t) &= (\mu-r)(t-t_0)+\frac{1}{2}V_0(t_0,t)+\sigma\sum_{j=1}^{\infty}\left(\psi_j\left(W^P_{t-jh}-W^P_{t_0-jh}\right)\right)+\sigma\sum_{i=0}^{0}\left(\psi_j\left(W^P_{t_0}-W^P_{t_0-ih}\right)\right) \\ &= \left(\mu-r+\frac{1}{2}\sigma^2\right)(t-t_0)+\sigma\left(W^P_{t_0}-W^P_{t_0-ih}\right)\sigma\sum_{j=1}^{\infty}\left(\psi_j\left(W^P_{t-jh}-W^P_{t_0-jh}\right)\right), \forall t\in[t_n,t_{n+1}). \end{aligned} \tag{C.6}$$

In view of Eq. (C.6), $A_0(t_0,t)$ is $\Im_{t_0}$-measurable and satisfies $A_0(t_0,t)<\infty$ because $\sum_{j=0}^{\infty}\left|\psi_j\right|$ and the realized paths of stock price and Brownian motion prior time t_0 are $\Im_{t_0}$-measurable and are assumed to be bounded. In view of Eq. (C.5), the definite integral $\int_{t_0}^{t}\varphi(s)ds<\infty$ holds for $t\in[t_0,t_1)$, which implies that $\varphi(s)<\infty$ holds for $s\in[t_0,t_1)$. Accordingly, the definite integral $\int_{t_0}^{t}(\varphi(s))^2ds<\infty$ holds for $t\in[t_1,t_2)$.

Similarly, for the case of $n=1$, conditioning on $\Im_{t_0}$, as the realized paths of stock price and the Brownian motion prior to time t_0 are bounded and $\Im_{t_0}$-measurable, $A_1(t_0,t)$ is $\Im_{t_0}$-measurable and satisfies $A_0(t_0,t)<\infty$ for $t\in[t_1,t_2)$. Therefore, Eq. (C.3) can be rewritten as

$$\begin{aligned} &A_1(t_0,t)+\sigma\sum_{i=0}^{1}\left(\psi_i\int_{t_0}^{t-ih}\varphi(s)ds\right) \\ &\quad= A_1(t_0,t)+\sigma\left(\int_{t_1}^{t}\varphi(s)ds+\int_{t_0}^{t_1}\varphi(s)ds\right)+\sigma\psi_1\int_{t_0}^{t-h}\varphi(s)ds=0, \forall t\in[t_1,t_2). \end{aligned} \tag{C.7}$$

Because $A_1(t_0,t)<\infty$ holds for $t\in[t_1,t_2)$ and $\int_{t_0}^{u}\varphi(s)ds<\infty$ holds for $u\in[t_0,t_1)$, we have the definite integral $\int_{tk}^{t}\varphi(s)ds<\infty$ for $t\in[t_k,t_{k+1})$, which implies that $\varphi(s)<\infty$ also holds for $s\in[t_1,t_2)$. Consequently, $\int_{t_0}^{t}(\varphi(s))^2ds=\int_{t_0}^{t_1}(\varphi(s))^2ds+\int_{t_1}^{t}(\varphi(s))^2ds<\infty$ holds for $t\in[t_1,t_2)$.

Assume that Eq. (C.4) be valid for $n=k-1$, which implies that $\varphi(s)<\infty$ holds for $s\in[t_0,t_k)$ and $\int_{t_0}^{t}(\varphi(s))^2ds<\infty$ for$t\in[t_{k-1,\,tk})$. For $n=k$, Eq. (C.3) is of the form:

$$A_k(t_0,t)+\left(\sigma\left(\int_{t_k}^{t}\varphi(s)ds+\int_{t_0}^{t_k}\varphi(s)ds\right)+\sigma\sum_{i=1}^{k}\left(\psi_i\int_{t_0}^{t-ih}\varphi(s)ds\right)\right)=0. \tag{C.8}$$

Because $A_k(t_0,t)<\infty$ holds for $t\in[t_k,t_{k+1})$ and $\int_{t_0}^{u}\varphi(s)ds$ holds for $u\in[t_0,t_k)$, based on Eq. (C.8), we have $\int_{t_k}^{t}\varphi(s)ds<\infty$ for $t\in[t_k,t_{k+1})$, which implies that $\varphi(s)<\infty$ holds for $s\in[t_k,t_{k+1})$. Thus,$\int_{t_0}^{t}(\varphi(s))^2ds=\int_{t_k}^{t}(\varphi(s))^2ds+\int_{t_0}^{t_k}(\varphi(s))^2ds<\infty$ holds for $t\in[t_k,t_{k+1})$. This completes the proof of Lemma 3.

Appendix D. The Proof of Theorem 1

To carry out the proof of Theorem 1, Eq. (20) is divided into two parts:

$$Ct_0=e^{-r(T-t_0)}W_Q[Max(S_T-K,0)|\Im_{t_0}]=A-B, \tag{D.1}$$

where

$$A=^{e-r(T-t_0)}E_Q\left[S_T1_{(S_T>K)}\middle|\Im_{t_0}\right]. \tag{D.2}$$

and

$$B=Ke^{-r(T-t_0)}E_Q\left[1_{(S_T>K)}\middle|\Im_{t_0}\right] \tag{D.3}$$

Under the risk neutral martingale measure Q, the stock price at time T equals

$$S_T=S_{t_0}\exp\left(r(T-t)+Z^Q_N(t_0,T)-\frac{1}{2}V_N(t_0,T)\right). \tag{D.4}$$

It is convenient to introduce an auxiliary probability measure Q^R on (Ω,F) by setting its Radon–Nikodym derivative as follows:

$$\xi^R_T=\frac{dQ_R}{d_Q}=\exp\left(Z^Q_N(t_0,T)-\frac{1}{2}V_N(t_0,T)\right). \tag{D.5}$$

By virtue of Eq. (14'), $Z_N^Q(t_0,T)$ satisfies

$$Z_N^Q(t_0,T) = 1_{(N>0)}\left(\sigma\sum_{j=0}^{N-1}\left(\left(\sum_{i=0}^{j}\psi_i\right)\left(W_{t-jh}^Q - W_{t-(j+1)h}^Q\right)\right)\right) + \sigma\left(\sum_{i=0}^{N}\psi_i\right)\left(W_{t-Nh}^Q - W_{t_0}^Q\right). \quad (14'')$$

It follows from Girsanov's theorem that the process W^R given by

$$W_v^R = W_v^Q - \sigma v\sum_{i=1}^{N}\psi_i, v\in[t_0, t-Nh), \quad (D.6)$$

$$W_v^R = W_v^Q - \sigma v\sum_{i=0}^{j}\psi_i, v\in[t-(j+1)h, t-jh)j = 0,\ldots,N-1. \quad (D.7)$$

is a standard Brownian motion under the probability measure Q^R. Therefore, $Z_n^Q(t_0,t)$ is equal to $Z_n^R(t_0,t)+V_n(t_0,T)$ and the dynamics of the stock price under the probability measure $Q^R$are

$$S_T = S_{t_0}\exp\left(r(T-t_0) + Z_N^R(t_0,T) + \frac{1}{2}V_N(t_0,T)\right). \quad (D.8)$$

Therefore, (D.2) can be rewritten as follows:

$$\begin{aligned} A &= S_{t_0}E_{Q_R}\left(1_{(S_T>K)}\middle|\mathfrak{I}_{t_0}\right) = S_{t_0}P_{rob}{}^{Q_R}\left(S_T > K\middle|\mathfrak{I}_{t_0}\right) \\ &= S_{t_0}P_{rob}{}^{Q_R}\left(\ln S_{t_0} + r(T-t_0) + Z_N^R(t_0,T) + \frac{1}{2}V_n(t_0,T) > \ln K\middle|\mathfrak{I}_{t_0}\right) \\ &= S_{t_0}P_{rob}{}^{Q_R}\left(\frac{\ln\frac{S_{t_0}}{K} + r(T-t_0) + \frac{1}{2}V_n(t_0,T)}{\sqrt{V_n(t_0,T)}} \geq \frac{-Z_N^R(t_0,T)}{\sqrt{V_N(t_0,T)}}\right) = S_{t_0}\Phi(d_{1N}(t_0,T)). \end{aligned} \quad (D.9)$$

Eq. (D.3) can be proved along the similar pricing procedure. For the proof of Eq. (22), since the discounted stock price and the ARMA(p, q)-type European options are Q-martingale, Eq. (22) is derived by using the well-known put call parity $P_{t_0} = C_{t_0} + Ke^{-r(T-t_0)} - S_{t_0}$. This ends the derivation of Theorem 1.

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